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Theory of dissipative solitons in complex Ginzburg-Landau systems

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Using soliton amplitude and phase ansatzes, a theory is proposed for searching for stationary soliton solutions to the cubic-quintic complex Ginzburg-Landau (CGL) equation. For arbitrary combinations of system parameters, our approach allows the existence of dissipative solitons together with their specific soliton characteristics to be determined, and we demonstrate this explicitly for the case of a pulsed fiber laser system. The regimes of existence of dissipative solitons and their rules of evolution in a complicated five-dimensional parameter space are also analyzed. This work may open other research opportunities in diverse areas of nonlinear dynamics governed by the CGL equation, and may impact significantly on experimental design.

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Solitons are ubiquitous in nature, appearing in diverse systems such as shallow water waves, DNA excitations, matter waves in Bose-Einstein condensates, and ultrashort pulses (or laser beams) in nonlinear optics [1]. They are a universal phenomenon of self-trapped wave packets, exhibiting properties typically associated with particles. In a telecommunications context, this unique particlelike nature has been applied to the study of collisions of temporal solitons [2]. Parallelly, it is this nature that enables spatial or spatiotemporal solitons to be proposed to use in all-optical signal processing and logic [3].

Originally, the terminology soliton was reserved for a particular set of integrable solutions existing as a result of the delicate balance between dispersion (or diffraction) and nonlinearity. However, similar classes of stable self-sustained structures can be found for a wide range of physical scenarios (e.g., in systems far from equilibrium), and it has become customary to also refer to these as solitons in the contemporary optics nomenclature. As a new paradigm of nonlinear waves, solitons in real dissipative environments, known as dissipative solitons [4], are attracting a significant surge of research activities on their spatial and/or temporal complexity during the last two decades, particularly for systems modeled by the cubic (quintic) complex Ginzburg-Landau (CGL) equations [5–7]. For this class of solitons, it has been realized that a separate balance between gain and loss is essential for their pattern formation, apart from that between dispersion (or diffraction) and nonlinearity.

The dynamics of dissipative solitons is thus markedly different from that of conventional solitons. It depends drastically on the system parameters and may evolve stationarily, periodically, or even chaotically [8–13], competing against other rich behaviors such as fronts, sources, and sinks [14,15]. In the great majority of cases, dissipative solitons serve as attractors with fixed profiles independent of the choice of input characteristics [11–13], analogous to the recently identified behavior of optical similaritons in a self-similar laser amplifier [16].

Unfortunately, the cubic-quintic CGL equation governing this dynamics involves a complicated five-dimensional parameter space and thus the search for its soliton solutions, even when stationary, has generally resorted to extensive numerical simulations [11–13], apart from some special exceptions in which exact solutions can be found [9,15]. As such, one of the main challenging open frontiers of this field is to elucidate the subtle dissipative dynamics using effective approximate theories, in order to provide improved physical insight and obviate the need for numerical attempts. Recently, Skarka *et al.* solved the CGL model for such dynamics using the variational theory [17], with their predictions agreeing well with numerical simulations for small nonlinear gain, but deviating significantly for flat-top solitons for which the nonlinear gain is large. Similarly, Tsoy *et al.* proposed a dynamical model for prediction of dissipative solitons using the moment method [18], but the results approximate well only for a portion of soliton regime. Quite expectedly, however, the limited range of validity of these theories originates in that they suppose an amplitude or a phase ansatz that explains only a small subset of rich behaviors of stationary dissipative solitons.

In this Rapid Communication, we propose a more complete theory for dissipative solitons in CGL systems suitable for a significantly wider parameter range compared to existing theories. By supposing a more general form of ansatz solution, we can offer an accurate estimate of the dynamics of stationary solitons such as their soliton characteristics, regimes of existence, and rules of evolution. Without loss of generality we illustrate our theory focussing on the case of a fiber laser, with its dynamics modeled by the cubic-quintic CGL equation [9–13,17,18]

$$i\psi_z + \tfrac{1}{2}\psi_{\tau\tau} + |\psi|^2\psi + \nu|\psi|^4\psi = i\delta\psi + i\epsilon|\psi|^2\psi + i\beta\psi_{\tau\tau} + i\mu|\psi|^4\psi, \tag{1}$$

where ψ denotes the normalized envelope of the field, z is the distance that the pulse travels, τ is the retarded time, δ is the linear gain-loss coefficient, β describes the spectral filtering, ϵ accounts for the nonlinear gain which arises from saturable absorption, the term with μ represents, if negative, the saturation of the nonlinear gain, while the one with ν corresponds, also if negative, to the saturation of the nonlinear refractive index.

For our purposes, we proceed to search for the stationary solutions of Eq. (1) of the form

$$\psi(z,\tau) = A(\tau)\exp[i\Phi(\tau) + i\Omega z], \tag{2}$$

where Ω is a real constant. Substitution of Eq. (2) into Eq. (1) followed by a separation of the real and imaginary terms yields two coupled ordinary differential equations. Since there are, apart from particular sets of system parameters [9,15], no exact analytical solutions in terms of elementary functions for such two equations, we use the amplitude and phase ansatzes

$$A(\tau) = \frac{\sqrt{P}}{\left[1 + (1-\eta)\sinh^2\left(\frac{\tau}{T}\right)\right]^{1/2}}, \tag{3}$$

$$\Phi(\tau) = d \ln A(\tau) + \mathcal{C}\int^{\tau} \tau' A^2(\tau')d\tau', \tag{4}$$

where P is the peak power of dissipative solitons, T is the pulse width, η denotes the shape factor with $0 \leq \eta \leq 1$, and $\mathcal{C}$ and d in combination account for the nonlinear pulse chirp given by $\delta\omega(\tau) = -d\Phi(\tau)/d\tau$. By inserting the ansatzes (3) and (4) into the intermediate differential equations and after some manipulations, one can obtain the reduced system of nonlinear equations

$$\delta + \epsilon P\alpha_1 + \frac{1}{2}\mu P^2\alpha_2 = \frac{\beta}{12}\mathcal{C}^2P^2T^2\alpha_5 - \frac{\beta}{2}\mathcal{C}dP\alpha_1 + \beta\sqrt{\frac{\delta^2+\Omega^2}{2\mathcal{B}}}\alpha_6, \tag{5}$$

$$(2\beta\epsilon+1)P(2\alpha_1\alpha_4 - \alpha_5) + (2\beta\mu+\nu)P^2(\alpha_2\alpha_4 - \alpha_1\alpha_5) = (2\beta\delta - \Omega)(\alpha_3\alpha_5 - 2\alpha_4) - \mathcal{B}\mathcal{C}dP(\alpha_1\alpha_4 - \alpha_5) + \sqrt{2\mathcal{B}(\delta^2+\Omega^2)}(\alpha_4\alpha_6 - \kappa\alpha_5), \tag{6}$$

where

$$\Omega = -\frac{\delta}{2\beta} - \frac{\epsilon - 2\beta}{2\beta}P\alpha_1 - \frac{\mu - 2\beta\nu}{4\beta}P^2\alpha_2, \tag{7}$$

$$\mathcal{C} = \frac{\epsilon - 2\beta}{\mathcal{B}}(\alpha_1\alpha_3 - 1) + \frac{\mu - 2\beta\nu}{2\mathcal{B}}P(\alpha_2\alpha_3 - 2\alpha_1), \tag{8}$$

$$d = \frac{2\beta\delta - \Omega + \sqrt{2\mathcal{B}(\delta^2+\Omega^2)}}{\delta + 2\beta\Omega}, \tag{9}$$

$$T = \sqrt{2\mathcal{B}}\sqrt{\frac{2\beta\delta - \Omega + \sqrt{2\mathcal{B}(\delta^2+\Omega^2)}}{(\delta+2\beta\Omega)^2}}. \tag{10}$$

Here $\mathcal{B} = 2\beta^2 + 1/2$ and the parameters α_i $(i=1,\ldots,6)$ are unique functions of η, given by $\alpha_1 = \frac{1}{2}\eta^{-1}[(1+\eta) - 2\kappa]$, $\alpha_2 = \frac{1}{2}\eta^{-1}[3(1+\eta)\alpha_1 - 2]$, $\alpha_3 = \frac{1}{2}(1+\eta) + 2\kappa$, $\alpha_4 = \frac{1}{2}\ell\alpha_1 - \kappa^{-1}$, $\alpha_5 = \frac{1}{2}\ell\alpha_2 - 3\kappa^{-1}\alpha_1$, and $\alpha_6 = 1 - \frac{1}{2}(1+\eta)\alpha_1$ with $\ell = \pi^2 + 4\,\mathrm{arctanh}^2(\sqrt{\eta})$ and $\kappa = \sqrt{\eta}/2\,\mathrm{arctanh}(\sqrt{\eta})$.

Equations (5)–(10) are the central theoretical results describing the dynamics of a wide range of stationary dissipative solitons. We point out that Eqs. (5) and (7) can follow from the variations with respect to the phase, Ωz, and the pulse energy, $E = \int_{-\infty}^{\infty}|\psi|^2 d\tau = PT/\kappa$, respectively [10,17], while Eqs. (9) and (10) result directly from the boundary conditions at $\tau = \pm\infty$. The remaining two equations are the natural results of integrating the intermediate differential equations over time from $-\infty$ to ∞, with the aid of Eqs. (9) and (10). It is easily seen that the particular sets of exact soliton solutions reported in Ref. [9] can be reproduced here. Take the cubic case for example, and it follows from Eq. (9) that $d = (\sqrt{2\mathcal{B}} - 1)/2\beta$ for $\delta = 0$. By substituting this together with Eqs. (7), (8), and (10) into Eqs. (5) and (6), we finally obtain that $\eta = 0$, $\mathcal{C} = 0$, $\Omega = (2\beta - \epsilon)P/3\beta$, and $T = \sqrt{\mathcal{B}d/\beta\Omega}$, leading to a one-parameter soliton family $A = \sqrt{P}\,\mathrm{sech}(\tau/T)$ with the parametric relation $\epsilon = \beta(3\sqrt{2\mathcal{B}} - 1)/2(2+9\beta^2)$, all as identical as in Ref. [9]. Nonetheless, we should stress that in most cases our system of equations is an otherwise approximate substitute for Eq. (1), but it exhibits an accuracy significantly better than existing theories and a utility suitable for interpretation and design of experiments, with errors comparable to the prediction of pulse propagation in an optical amplifier using a self-similar parabolic ansatz [19].

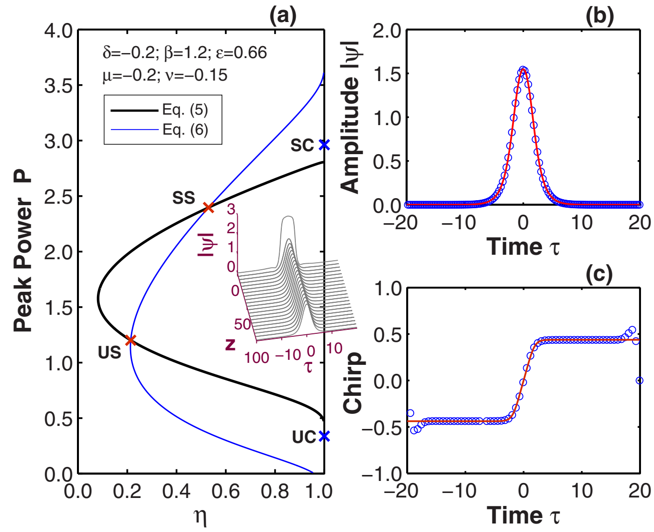

FIG. 1. (Color online) (a) Fixed points with red crosslets denoting a stable soliton (SS) and an unstable soliton (US) given by Eqs. (5) and (6) for a set of system parameters specified in the figure. The evolution of an arbitrary input pulse towards a stationary dissipative soliton is shown in the inset. The stable cw (SC) and unstable cw (UC) are denoted by blue crosslets at $\eta = 1$. (b) and (c) show, respectively, the amplitude and chirp of the stable soliton, related to the characteristics $P = 2.3970$, $\eta = 0.5300$, $T = 1.2367$, $\mathcal{C} = -0.0235$, and $d = 0.5418$. Solid curves: Predicted results; Open circles: Simulated results.

From Eqs. (5) and (6), one can determine whether a dissipative soliton exists definitely or not for arbitrary combinations of system parameters and, if so, one can determine accurately all of its pulse features such as the peak power P, the pulse width T, the shape characterized by η, and the pulse chirp given by $\mathcal{C}$ and d. We illustrate this in Figs. 1 and 2 by choosing two different sets of system parameters and we compare our predictions with numerical simulations. For a given set of five parameters, two possible soliton solutions (red crosses) can be found by solving the coupled equations (5) and (6), which are, respectively, depicted as the thick and thin lines in Figs. 1(a) and 2(a). It is clear that according to previous linear stability analyses [4] and Lyapunov exponent

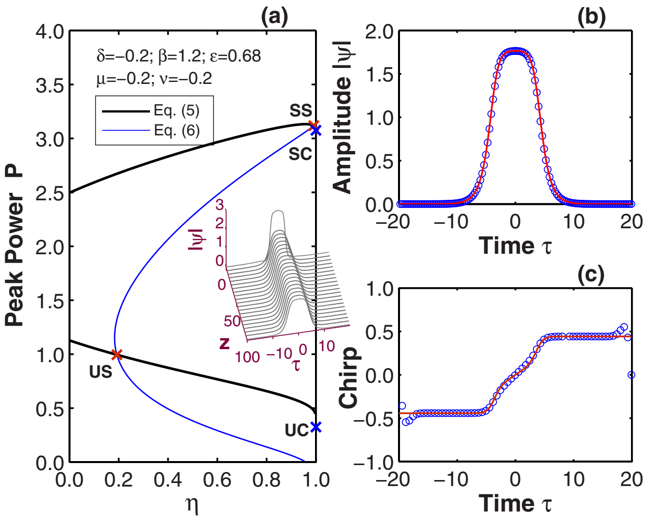

FIG. 2. (Color online) The same as in Fig. 1 except for using another set of system parameters specified in (a), which yields a stable flat-top soliton with the characteristics $P=3.1181$, $\eta=0.9916$, $T=1.2285$, $\mathcal{C}=-0.0131$, and $d=0.5436$.

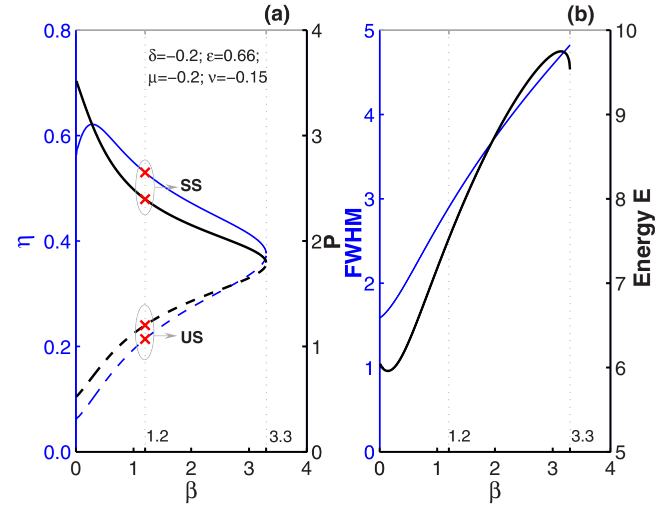

FIG. 3. (Color online) Evolutions of soliton characteristics: (a) η (blue) and P (black); (b) FWHM (blue) and energy E (black), with the parameter β. Here solid and dashed curves stand for the stable and unstable solitons, respectively, and crosses in (a) for the SS and US points in Fig. 1.

calculations [17], the soliton with larger peak power, denoted by SS, is definitely stable while the smaller one, US, is unstable. Independently of its initial pulse energy and profile, an input pulse always evolves towards the fixed point SS, just as indicated by the pulse evolutions in Figs. 1(a) and 2(a). This is a typical feature of attractors and was extensively discussed before [4].

Hence, by inserting the values of P and η into Eqs. (7)–(10), the other soliton characteristics Ω, $\mathcal{C}$, d, T, and as a result the amplitude (3) and phase (4) follow easily. It is striking that the predicted amplitude, $A(\tau)$, and pulse chirp, $\delta\omega(\tau)$, are in excellent agreement with numerical simulations, both for the Gaussian-type soliton [Figs. 1(b) and 1(c)] and for the flat-top one [Figs. 2(b) and 2(c)]. Quantitatively, we introduce a quantity $\Delta\psi=(\int|\psi-\psi^b|^2 d\tau)^{1/2}/(\int|\psi|^2 d\tau)^{1/2}$ for a measure of deviation, where ψ and ψ^b are the numerical and predicted solutions, respectively, and show by excluding the uncontrollable wave number effect that $\Delta\psi$ has a value of 0.7% for the former case and 3% for the latter one, both indicating sufficiently good accuracy of analytical predictions. Particularly, additional simulations show that errors of <1% can be obtained for parameters typical of current experimental systems (refer to Chaps. 7 and 9 in Ref. [4], for example) and thus the analytical approach presented above is well adapted to both the interpretation of experimental results and the future optimization of related experimental systems.

Additionally, it is of interest to note that as $\eta=1$, Eqs. (5) and (6) can be reduced to the same quadratic equation $\delta+\epsilon P+\mu P^2=0$, yielding two continuous wave (cw) solutions with $\mathcal{C}=0$ [18]. For comparison, we depict such two cw solutions by blue crosses in Figs. 1(a) and 2(a). They are, respectively, the asymptotes of two soliton solutions, SS and US, for η approaching to unity, and thereby the higher cw, SC, is stable while the lower one, UC, is unstable. An inspection of Figs. 1 and 2 reveals that as ϵ increases, the soliton shape evolves from a Gaussian-like shape ($0\leqslant\eta<0.9$) into a flat-top shape ($0.9<\eta<1$), and eventually broadens to be a stable cw for a certain larger ϵ ($\eta=1$). We stress that the stable cw here also serves as a attractor such that any small localized excitation can evolve into it, with the leading and trailing edges behaving as two moving fronts.

More significantly, we can predict the regimes of existence of dissipative solitons in the five-dimensional parameter space. For simplicity, we only consider here the change of soliton characteristics with one parameter at a time around the fixed point SS in Fig. 1, keeping the other four parameters identically unchanged. Figure 3 shows the evolutions of the shape factor η, the peak power P, the full width at half-

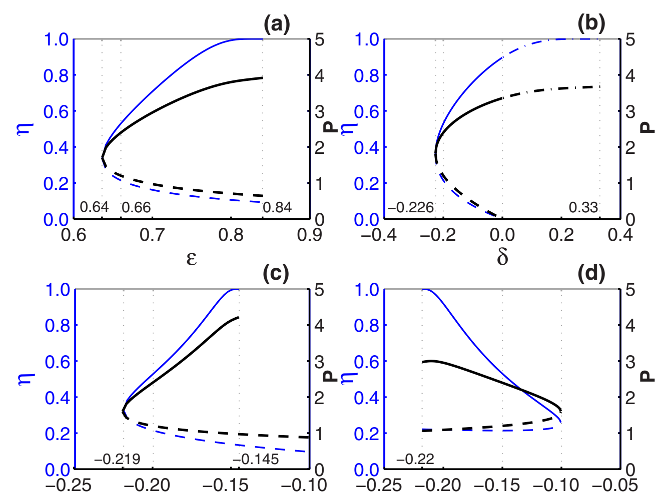

FIG. 4. (Color online) Evolutions of the shape factor η (blue) and peak power P (black) versus (a) ϵ, (b) δ, (c) μ, and (d) ν under otherwise identical conditions as in Fig. 1. Here solid, dashed, and dashed-dotted curves denote the stable, unstable, and metastable solitons, respectively, with their regimes of existence separated by the gray broken vertical lines.

maximum (FWHM) given by $T_{\mathrm{FWHM}}=2T\,\mathrm{arcsinh}(1/\sqrt{1-\eta})$, and the pulse energy E with the spectral parameter β under otherwise existing conditions. In Fig. 3(a) we also display the evolution of unstable soliton with β using dashed curves, along with an indication of the fixed points SS and US at $\beta=1.2$ by crosses. It is clearly shown in these double labeling plots that the stable stationary soliton exists only in an interval of β (here in between 0 and 3.3), and moreover, as β grows, the soliton peak power is monotone decreasing while its FWHM increases, resulting in an increase of pulse energy except for a slight decrease at both ends. Intriguingly, the shape factor η reaches its maximum first (here at $\beta=0.29$) and then decreases into its minimum at the end of the interval. This implies that the spectral filtering tends to broaden and lower the pulse.

The other rules of soliton characteristics evolving with ϵ, δ, μ, and ν are depicted seriatim in the subplots in Fig. 4, all around the fixed point SS in Fig. 1. As can be clearly seen, the existence of stable solitons is limited by two broken vertical lines located around the solid ones, e.g., in between 0.64 and 0.84 for ϵ. Further, it is shown that both the peak power and the shape factor for the stable soliton increase with the parameters ϵ, δ, and μ, but decrease with ν. Apparently, for a certain value of ϵ, δ, μ, or ν, the factor η arrives at unity and as a result the soliton evolves into a stable cw. A comparison of Fig. 4(a) with Fig. 3 in Ref. [17] shows that our result about the peak power is significantly better for prediction in a significantly wider ϵ region. Besides, we depict by dashed-dotted curves in Fig. 4(b) the evolutions of P and η with the gain from 0 to 0.33. In such an interval, the dissipative soliton is found to be metastable and may, after a long propagation distance, evolve into a multisoliton bound state [10] or into a stable cw due to an instability of the background [4]. All of these unexpected dynamics have been corroborated by extensive simulations.

In conclusion, in order to establish the theoretical framework of stationary dissipative solitons governed by the cubic-quintic CGL equation, we have developed an analytical approach using amplitude and phase ansatzes. With this theory, one can predict accurately the possibility of existence of dissipative solitons in the underlying five-dimensional parameter space, can determine their amplitude, shape, width, chirp, and even their regimes of existence, and can find the rules of evolution of soliton characteristics with the physical system parameters. Based on these results, it is possible to construct a desired dissipative soliton without time-consuming numerical attempts. In view of the increasing importance of the CGL equation in description of highly diverse nonlinear phenomena [7], we anticipate that our results will open new research opportunities and may result in a substantial impact on experiments. Particularly, in nonlinear optics, our theory may have practical implications in such active areas as the design of passively mode-locked lasers [4,13] and the study of soliton propagation in photonic crystal fibers with gain and filtering [20].

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